Characterizing complexity classes with boolean theories extensions

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Descriptive complexity : languages and computations

Logical description

- Finite structures over finite signatures
- Logical ressources (higher order quantifiers, operators)

Decision algorithm

- Models of computations (Turing machines, circuits)
- Computational ressources (Time, space)

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Descriptive complexity

$$x \models F \iff x \in L$$

Ex : Palindromes

Our approach what's different ?

~~Logical description~~ Boolean theories

- Finite models of first order finite theories (adding axioms to signatures)
- Logical ressources ~~(higher order quantifiers, operators)~~ sorts, relations

Boolean Theories

Definition

A **Boolean theory** $\mathbb{T}$ is a triple

$$(\mathrm{Sort}(\mathbb{T}), \mathrm{Rel}(\mathbb{T}), \mathrm{Ax}(\mathbb{T}))$$

A Boolean theory $\mathbb{T}$ is **finite** if $\mathrm{Sort}(\mathbb{T})$, $\mathrm{Rel}(\mathbb{T})$ and $\mathrm{Ax}(\mathbb{T})$ are all finite.

Example : $\mathbb{S}\mathrm{tr}$

Definition

$$\mathrm{Sort}(\mathbb{S}\mathrm{tr}) = \{N\}$$
$$\mathrm{Rel}(\mathbb{S}\mathrm{tr}) = \{\leq \rightarrowtail N \times N, \mathrm{isOne} \rightarrowtail N\}$$
$$\mathrm{Ax}(\mathbb{S}\mathrm{tr}) = \{\text{“} \leq \text{is a total order''}\}$$

$$\begin{array}{cccccccccc}
(N, \leq) & : & \mathsf{a} & \leq & \mathsf{b} & \leq & \mathsf{c} & \leq & \mathsf{d} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{isOne} & : & 0 & & 1 & & 0 & & 1
\end{array}$$

Other example : $\mathbb{G}\mathrm{rph}$

Definition

$$\begin{aligned}
\mathrm{Sort}(\mathbb{G}\mathrm{rph}) &= \{V\} \\
\mathrm{Rel}(\mathbb{G}\mathrm{rph}) &= \{E \rightarrowtail V \times V\} \\
\mathrm{Ax}(\mathbb{G}\mathrm{rph}) &= \emptyset
\end{aligned}$$

Extension of a theory

Definition

$\mathbb{T}$ extends $\mathbb{T}'$ iff :

- $\mathrm{Sort}(\mathbb{T}') \subseteq \mathrm{Sort}(\mathbb{T})$
- $\mathrm{Rel}(\mathbb{T}') \subseteq \mathrm{Rel}(\mathbb{T})$
- $\mathrm{Ax}(\mathbb{T}') \subseteq \mathrm{Ax}(\mathbb{T})$

Definition

$\mathbb{T}$ is a relational extension of $\mathbb{T}'$ iff :

- $\mathbb{T}'$ is an extension of $\mathbb{T}$
- $\mathrm{Sort}(\mathbb{T}') = \mathrm{Sort}(\mathbb{T})$

What is different ? What isn't ?

~~What is different ?~~ What isn't ?

Fagin's Theorem (our version)

Theorem (Fagin (Boolean sauce))

NP is equal to the relational extensions of $\mathbb{S}\mathrm{tr}$.

Sketch of the proof

Logical description

Given a NP Turing machine :

- Give a theory such that all its finite models can project to accepting runs of the machine
- Is this a relational extension of $\mathbb{S}\mathrm{tr}$? (without detail)

Decision algorithm

Given a relational extension of $\mathbb{S}\mathrm{tr}$

- Give a Turing machine whose accepting runs are models of the theory
- Is this a NP Turing machine?

Extending strings with table of symbols

$$\mathbb{S}\mathrm{tr} + \textbf{S,T} + \mathrm{Symb}_0, \mathrm{Symb}_1, \mathrm{Symb}_\square \rightarrowtail T \times S + (\mathrm{State}_q) \rightarrowtail T$$

Axioms :

- **S, T** are finite chains (equipped with successors and max)
- $\mathrm{Symb}_{\{0,1,\square\}}$ form a function from $T \times S$ to $\{0, 1, \square\}$ and $(State_q)$ from T to Q
- State q_0 and blank symbols $\square$ on work tape at time 0. State $accept$ at final state

i_0	i_1	i_2		i_{n-1}	i_n
0	0	1	...	0	0

	s_1	s_2	...	s_{n-1}	s_n	
t_1	1	1	...	0	0	q_0
t_2	0	1	...	0	1	q
$\vdots$			...			$\vdots$
t_{m-1}	1	0	...	0	0	q'
t_m	1	1	...	0	0	$accept$

Adding heads

$$\mathbb{S}\mathrm{tr} + \textbf{S,T} + \mathrm{Symb}_{\{0,1,\square\}}, (\mathrm{State}_q) + \mathrm{wHead} \rightarrowtail T \times S + \mathrm{iHead} \rightarrowtail T \times N$$

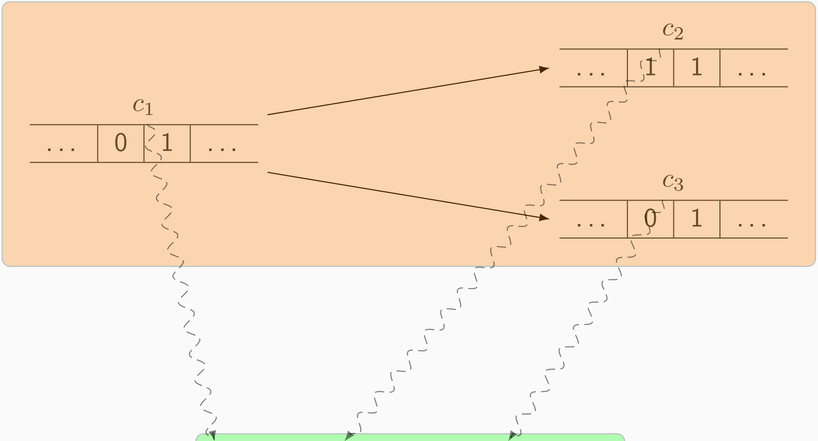
Axioms :

- wHead (resp. iHead) are functions from T to S (resp. N)
- wHead and iHead don't move more than one case
- The work tape is unchanged at positions where the head is not found

i_0	i_1	i_2		i_{n-1}	i_n
0	0	1	...	0	0

	s_1	s_2	$\dots$	s_{n-1}	s_n	
t_1	1	1	...	0	0	q_0
t_2	0	1	...	0	0	q
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$
t_{m-1}	1	0	...	0	0	q'
t_m	1	1	...	0	1	$accept$

Axioms for the transitions



$$\Phi_{c_1}(t) \;\Rightarrow\; \Phi_{c_2}(t+1) \;\vee\; \Phi_{c_3}(t+1)$$

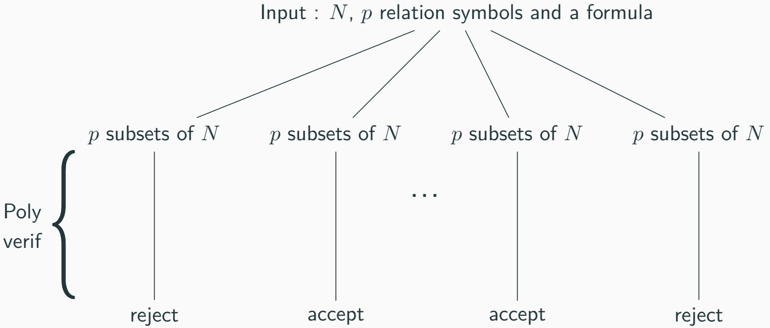
Polynomiality

All Turing mahines are represented !

But $\#T$ is exactly the time of the Turing machine

Computing a model of a theory

A drawing worths more than a thousand word



Example : $\mathbb{G}\mathrm{rph}$ and 3-colorability

$$\mathbb{G}\mathrm{rph} + C_1, C_2, C_3 + Ax$$

Axioms :

- $C_1 \vee C_2 \vee C_3$
- For $i \neq j, C_i \wedge C_j \Rightarrow \perp$
- $E(x, y) \wedge C_i(x) \wedge C_j(y) \Rightarrow i \neq j$

Further results

- Polynomial Hierarchy
- PSPACE
- NL
- Logarithmic Hierarchy